

# Spin liquids on a honeycomb lattice: Projective Symmetry Group study of Schwinger fermion mean-field theory

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Spin liquids are novel states of matter with fractionalized excitations. A recent numerical study of Hubbard model on a honeycomb lattice<sup>1</sup> indicates that a gapped spin liquid phase exists close to the Mott transition. Using Projective Symmetry Group, we classify all the possible spin liquid states by Schwinger fermion mean-field approach. We find there is only one fully gapped spin liquid candidate state: “Sublattice Pairing State” that can be realized up to the 3rd neighbor mean-field amplitudes, and is in the neighborhood of the Mott transition. We propose this state as the spin liquid phase discovered in the numerical work. To understand whether SPS can be realized in the Hubbard model, we study the mean-field phase diagram in the  $J_1 - J_2$  spin-1/2 model and find an  $s$ -wave pairing state. We argue that  $s$ -wave pairing state is not a stable phase and the true ground state may be SPS. A scenario of a continuous phase transition from SPS to the semimetal phase is proposed. This work also provides guideline for future variational studies of Gutzwiller projected wavefunctions.

## I. INTRODUCTION

Traditional Landau’s theory<sup>2,3</sup> points out that states of matter can be classified by their symmetry. And the low energy excitations can be understood by either bosonic modes or fermionic quasiparticles, which carry integer multiples of the quantum numbers of the fundamental degrees of freedom. Fractional quantum Hall liquids (FQHLs) provide a striking counterexample of the Landau’s paradigm: different FQHLs all have the same symmetry, yet they are very different since a quantum phase transition is required to go from one liquid to another. To understand their differences, one has to go beyond Landau’s paradigm and the concept of topological order was introduced<sup>4,5</sup>. The quasiparticle excitations in FQHLs carry only a fraction of the fundamental electric charge. Meanwhile these fractionalized quasiparticles obey neither bosonic nor fermionic statistics and are dubbed anyons consequently.

Can strong interactions lead to similar novel states of matter in the absence of magnetic field? After the original proposal of Anderson<sup>6</sup>, intensive theoretical studies have revealed that spin systems can realize such novel phases of matter: spin liquids (SL). And a few experimental systems have been identified to be likely in spin liquid phases<sup>7-9</sup>. A spin liquid is often defined to be a quantum phase of spin-1/2 per unit cell that does not break translation symmetry. These liquid phases of spins are also distinct from one another by their topological order. Although a rigorous theorem is lacking because we are still unable to classify all possible topological order, it is generally believed that the excitation of a topological ordered phase is fractionalized<sup>10</sup>.

Although theoretical studies have shown that spin liquid ground states exist for artificial model Hamiltonians<sup>11-16</sup>, it remains unclear whether a simple or experimentally realizable Hamiltonian can host such novel states. Recently a remarkable quantum Monte

Carlo simulation of Hubbard model on a honeycomb lattice<sup>1</sup> indicates that a gapped spin disordered ground state exists in the neighborhood of the Mott transition. Although a honeycomb lattice has two spin-1/2 per unit cell, it is impossible to have a band insulator phase without breaking lattice symmetry. Therefore this spin disordered phase should be topologically ordered and have fractionalized excitations. We will still term it a spin liquid.

What is the nature of this spin liquid phase? In this paper we try to propose the candidate states by Schwinger-fermion (or slave-boson) mean-field approach<sup>11,17-22</sup>, following the techniques developed in Ref. 23. Our results can be summarized as follows. We search for the fully gapped spin liquids which lead us to focus on the  $Z_2$  mean-field states. We first use Projective Symmetry Group (PSG)<sup>23</sup> to classify all 128 possible  $Z_2$  mean-field states that preserve the full lattice symmetry as well as time-reversal symmetry. Notice the spin liquid phase in the numerical work seems to be connected to the semimetal phase by a second-order phase transition, which suggests this state to be in the neighborhood of a uniform Resonating-Valence-Bond (u-RVB) state. So we classify all the 24 possible  $Z_2$  states around the u-RVB states. Among these 24 states, we find only 4 states can have a full energy gap in the spinon spectrum, while other 20 states have symmetry protected gapless spinon excitations. We find that up to 3rd neighbor mean-field amplitudes, only one of the four fully gapped  $Z_2$  state can be realized, and we term it as Sublattice Pairing State (SPS). We propose this state to be the spin liquid state discovered in the numerical study. We also study the mean-field phase diagram of the  $J_1 - J_2$  antiferromagnetic spin-1/2 model on a honeycomb lattice to understand whether SPS can be more favorable than the u-RVB state while both states are in the neighborhood of the Mott transition. We find when  $J_2 > 0.85J_1$  a spinon gap opens up by  $s$ -wave pairing on top of the u-RVB state. This  $s$ -wave pairing state is not a stable

phase and is an artifact of the mean-field study where gauge dynamics are ignored. On the other hand, the proposed SPS  $Z_2$  state is continuously connected to the  $s$ -wave pairing state by making the pairing phase sublattice dependent. This suggests the ultimate fate of  $s$ -wave pairing state may be SPS. We propose that a more careful projected wavefunction study, which includes the gauge fluctuations, may be able to find SPS  $Z_2$  state as the ground state in the  $J_1 - J_2$  model. The possible continuous phase transitions from SPS into semi-metal phase are discussed.

## II. SCHWINGER-FERMION APPROACH AND PSG

In Schwinger-fermion approach, a spin-1/2 operator at site  $i$  is represented by:

$$\vec{S}_i = \frac{1}{2} f_{i\alpha}^\dagger \vec{\sigma}_{\alpha\beta} f_{i\beta}. \quad (1)$$

A Heisenberg spin Hamiltonian  $H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j$  is represented as  $H = \sum_{\langle ij \rangle} -\frac{1}{2} J_{ij} (f_{i\alpha}^\dagger f_{j\alpha} f_{j\beta}^\dagger f_{i\beta} + \frac{1}{2} f_{i\alpha}^\dagger f_{i\alpha} f_{j\beta}^\dagger f_{j\beta})$ . Because this representation enlarges the Hilbert space, states need to be constrained in the physical Hilbert space, i.e., one  $f$ -fermion per site:

$$f_{i\alpha}^\dagger f_{i\alpha} = 1, \quad f_{i\alpha} f_{i\beta} \epsilon_{\alpha\beta} = 0. \quad (2)$$

Introducing mean-field parameters  $\eta_{ij} \epsilon_{\alpha\beta} = -2 \langle f_{i\alpha} f_{j\beta} \rangle$ ,  $\chi_{ij} \delta_{\alpha\beta} = 2 \langle f_{i\alpha}^\dagger f_{j\beta} \rangle$ , where  $\epsilon_{\alpha\beta}$  is fully antisymmetric tensor, after Hubbard-Stratonovich transformation, the Lagrangian of the spin system can be written as<sup>23</sup>

$$L = \sum_i \psi_i^\dagger \partial_\tau \psi_i + \sum_{\langle ij \rangle} \frac{3}{8} J_{ij} \left[ \frac{1}{2} \text{Tr}(U_{ij}^\dagger U_{ij}) - (\psi_i^\dagger U_{ij} \psi_j + h.c.) \right] + \sum_i a_0^l(i) \psi_i^\dagger \tau^l \psi_i \quad (3)$$

where two-component fermion notation  $\psi_i = (f_{i,\uparrow}, f_{i,\downarrow})^T$  is introduced for reasons that will be explained shortly.  $U_{ij}$  is a matrix of mean-field amplitudes:

$$U_{ij} = \begin{pmatrix} \chi_{ij}^\dagger & \eta_{ij} \\ \eta_{ij}^\dagger & -\chi_{ij} \end{pmatrix}. \quad (4)$$

$a_0^l(i)$  are the local Lagrangian multipliers that enforces the constraints Eq.(2).

In terms of  $\psi$ , Schwinger-fermion representation has an explicit  $SU(2)$  gauge redundancy: a transformation  $\psi_i \rightarrow W_i \psi_i$ ,  $U_{ij} \rightarrow W_i U_{ij} W_j^\dagger$ ,  $W_i \in SU(2)$  leaves the action invariant. This redundancy is originated from representation Eq.(1): this local  $SU(2)$  transformation leaves the spin operators invariant<sup>20</sup> and thus does not change physical Hilbert space.

One can try to solve Eq.(3) by mean-field (or saddle-point) approximation. At mean-field level,  $U_{ij}$  and  $a_0^l$

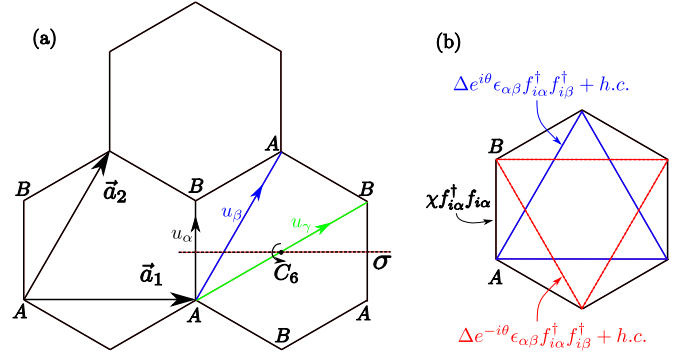


FIG. 1: (color online) (a) Honeycomb lattice and generators of symmetry group. (b) SPS mean-field ansatz in terms of  $f$ -fermion.  $t, \Delta, \theta$  are real and  $\theta \neq 0, \pi, \pm\pi/2$ . The pairing phases for A sublattice (blue solid line) and B sublattice (red dashed line) are opposite.

are treated as complex numbers, and  $a_0^l$  must be chosen such that constraints Eq.(2) are satisfied at the mean field level:  $\langle \psi_i^\dagger \tau^l \psi_i \rangle = 0$ . The mean-field ansatz can be written as:

$$H_{MF} = - \sum_{\langle ij \rangle} \psi_i^\dagger u_{ij} \psi_j + \sum_i \psi_i^\dagger a_0^l \tau^l \psi_i. \quad (5)$$

where  $u_{ij} = \frac{3}{8} J_{ij} U_{ij}$ . A local  $SU(2)$  gauge transformation modify  $u_{ij} \rightarrow W_i u_{ij} W_j^\dagger$  but does not change the physical spin state described by the mean-field ansatz. By construction the mean-field amplitudes do not break spin rotation symmetry, and the mean field solutions describe spin liquid states if translational symmetry is preserved. Different  $\{u_{ij}\}$  ansatz may be in different spin liquid phases. The mathematical language to classify different spin liquid phases is PSG<sup>23</sup>.

PSG is the manifestation of topological order in the Schwinger-fermion representation: spin liquid states described by different PSG's are different phases. It is defined as the collection of all combinations of symmetry group and  $SU(2)$  gauge transformations that leave  $\{u_{ij}\}$  invariant (as  $a_0^l$  are determined self-consistently by  $\{u_{ij}\}$ , these transformations also leave  $a_0^l$  invariant). The invariance of a mean-field ansatz  $\{u_{ij}\}$  under an element of PSG  $G_U U$  can be written as

$$\begin{aligned} G_U U(\{u_{ij}\}) &= \{u_{ij}\}, \\ U(\{u_{ij}\}) &\equiv \{\tilde{u}_{ij} = u_{U^{-1}(i), U^{-1}(j)}\}, \\ G_U(\{u_{ij}\}) &\equiv \{\tilde{u}_{ij} = G_U(i) u_{ij} G_U(j)^\dagger\}, \\ G_U(i) &\in SU(2). \end{aligned} \quad (6)$$

Here  $U \in SG$  is an element of symmetry group (SG) of the spin liquid state. SG on a honeycomb lattice is generated by time reversal  $T$ , reflection  $\sigma$ ,  $\pi/3$  rotation  $C_6$  and translations  $T_1, T_2$  as illustrated in FIG. 1 (see also appendix A).  $G_U$  is the gauge transformation associated with  $U$  such that  $G_U U$  leaves  $\{u_{ij}\}$  invariant.

There is an important subgroup of PSG, Invariant Gauge Group (IGG), which is composed of all

the pure gauge transformations in PSG:  $IGG \equiv \{\{W_i\} | W_i u_{ij} W_j^\dagger = u_{ij}, W_i \in SU(2)\}$ . One can always choose a gauge in which the elements in IGG is site-independent. In this gauge, IGG can be global  $Z_2$  transformations:  $\{W_i = \tau^0, W_i = -\tau^0\}$ , the global  $U(1)$  transformations:  $\{W_i = e^{i\theta\tau^3}, \theta \in [0, 2\pi]\}$ , or the global  $SU(2)$  transformations:  $\{W_i = e^{i\theta\hat{n}\cdot\vec{\tau}}, \theta \in [0, 2\pi], \hat{n} \in S^2\}$ , and we dub them  $Z_2$ ,  $U(1)$  and  $SU(2)$  state respectively.

The importance of IGG is that it controls the low-energy gauge fluctuations. Beyond mean-field level, fluctuations of  $U_{ij}$  and  $a_0^l$  need to be considered and the mean-field state may or may not be stable. The low-energy effective theory is described by fermionic spinon band structure coupled with a dynamical gauge field of IGG. For example,  $Z_2$  state with gapped spinon dispersion can be a stable phase because the low-energy  $Z_2$  dynamical gauge field can be in the deconfined phase<sup>24,25</sup>. But for a  $U(1)$  state with gapped spinon dispersion, the  $U(1)$  gauge fluctuations would generally drive the system into confinement due to monopole proliferation<sup>26</sup>, and the mean-field state would be unstable. And an  $SU(2)$  state with gapped spinon dispersion should also be in the confined phase because there is no known IR stable fixed point of pure  $SU(2)$  gauge theory in 2+1 dimension. Because the purpose of this paper is to search for stable spin liquid phases that has a Schwinger fermion mean-field description, we will focus on  $Z_2$  states.

If  $G_U U \in PSG$  and  $g \in IGG$ , by definition we have  $gG_U U \in PSG$ . This means that the mapping  $h : PSG \rightarrow SG : f(G_U U) = U$  is a many-to-one mapping. In fact it is easy to show that mapping  $h$  induces group homomorphism<sup>23</sup>:

$$PSG/IGG = SG. \quad (7)$$

Mathematically  $PSG$  is an extension of  $SG$  by  $IGG$ .

Our definition of PSG requires a mean-field ansatz  $\{u_{ij}\}$ . With Eq.(7), one can define algebraic-PSG which does not require ansatz  $\{u_{ij}\}$ . An algebraic-PSG is simply defined as a group satisfying Eq.(7). Obviously a PSG (realizable by an ansatz) must be an algebraic-PSG, but the reverse may not be true, because sometimes an algebraic-PSG cannot be realized by any mean-field ansatz.

To classifying all possible  $Z_2$  Schwinger-fermion mean-field states, we need to find all possible  $PSG$  group extensions of the  $SG$  with a  $Z_2$  IGG. Here  $SG$  is the direct product of the space group of honeycomb lattice and the time-reversal  $Z_2$  group. In appendix A we show the general constraints that must be satisfied for such a group extension. In appendix B, using these constraints, we find there are in total 160  $Z_2$  algebraic-PSGs on honeycomb lattice. And at most 128 PSGs of them can be realized by an ansatz  $\{u_{ij}\}$ . These 128 PSGs are the complete classification of  $Z_2$  spin liquids on a honeycomb lattice.

### III. CLASSIFICATION OF $Z_2$ STATES AROUND THE U-RVB STATE

Can one further identify the candidate states for the spin liquid discovered in the numerical study<sup>1</sup>? The answer is yes. Numerically the spin liquid phase is found close to the Mott transition and it seems to be connected to the semimetal phase by a continuous phase transition. What are the  $Z_2$  Schwinger-fermion states in the neighborhood of the semi-metal phase?

Are there Schwinger-fermion mean-field states that can be connected to the semi-metal phase via a continuous phase transition? This question was firstly discussed by Hermele in Ref. 27. Using slave-rotor formalism, it was shown that the semi-metal phase can go through a continuous phase transition into an  $SU(2)$  u-RVB state (also termed as algebraic spin liquid (ASL) in Ref. 27) at the mean-field level. This  $SU(2)$  u-RVB ansatz, in terms of  $f$ -spinon, can be written as  $H_{MF} = t \sum_{\langle ij \rangle} f_{i\alpha}^\dagger f_{j\alpha}$ ,  $t$  is real and summation is over all nearest neighbor bond. The single-spinon dispersion of u-RVB state is similar to the electronic dispersion in the semi-metal phase, which is composed of four two-component Dirac cones at the corner of Brillouin Zone, two from spin and two from valley. Physically it is easy to understand u-RVB state connecting with the semi-metal phase: At the Mott transition, only the charge fluctuation becomes fully gapped and the spinon dispersion still remember the semi-metal band structure.

The u-RVB ansatz can be simply expressed as a graphene-like nearest neighbor hopping of  $f$ -fermions: Fig.1:

$$H_{MF}^{uRVB} = \chi \sum_{\langle ij \rangle} f_{i\alpha}^\dagger f_{j\alpha}, \quad (8)$$

where  $\chi$  is real. Beyond mean-field level, the low-energy effective theory of u-RVB state is described by  $N_f = 2$  two-component Dirac spinons ( $SU(2)$  gauge doublet) coupled with a dynamical  $SU(2)$  gauge field<sup>27</sup>, i.e. QCD<sub>3</sub>. In the large- $N_f$  limit QCD<sub>3</sub> has a stable IR fixed point with gapless excitations and can be a stable ASL phase<sup>28</sup>. When  $N_f = 0$  the pure gauge QCD<sub>3</sub> is in a confined phase<sup>29,30</sup>. This indicates a critical  $N_c$  and when  $N_f < N_c$  confinement occurs<sup>28</sup>. Although no controlled estimate of  $N_c$  is available, a self-consistent solution of the Schwinger-Dyson equations<sup>28</sup> suggests  $N_c \approx \frac{64}{\pi^2}$ . We will assume that  $N_c > 2$  and therefore u-RVB state is not a stable phase.

Due to the lack of the knowledge of the confinement mechanism, it is difficult to reliably predict the ultimate fate of the u-RVB state (or ASL). But one possibility is that the strong gauge interaction induces Higgs condensation which breaks the  $SU(2)$  gauge symmetry down to  $Z_2$ , so that the renormalization group flows into a stable fixed point of  $Z_2$  gauge theory. Based on this assumption, we can propose a scenario of a continuous phase transition from the semi-metal phase into a  $Z_2$  spin liq-

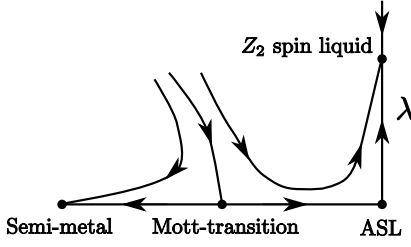


FIG. 2: Schematic RG flow of the Mott transition.  $\lambda$  represents a relevant perturbation of the ASL fixed point, which eventually drive the RG flow into a stable  $Z_2$  spin liquid fixed point.

uid phase: the critical point is still described by the slave-rotor critical theory discussed in Ref. 27. But on the Mott insulator side, a dangerously irrelevant operator (for example, can be a four-fermion interaction term) becomes relevant and finally drive the RG flow away from the ASL fixed point and flow into a stable fixed point of a  $Z_2$  phase by Higgs mechanism. Here we assume the ASL still describes an unstable fixed point with relevant directions. This scenario is schematically shown in Fig.2, which has the same spirit of the deconfined quantum criticality<sup>31</sup>.

If this scenario is correct, the mean-field ansatz of the  $Z_2$  spin liquid should be connected to the u-RVB ansatz by a continuous Higgs condensation, which breaks the  $SU(2)$  IGG down to  $Z_2$ . During this transition, the u-RVB ansatz  $\{u_{ij}^{uRVB}\} \rightarrow \{u_{ij}^{uRVB} + \delta u_{ij}\}$  and the  $\delta u_{ij}$  amplitudes play the role of the Higgs boson. We define a  $Z_2$  state to be around (or in the neighborhood of) the u-RVB when the  $Z_2$  state can be obtained by an infinitesimal change  $\{u_{ij}^{uRVB}\} \rightarrow \{u_{ij}^{uRVB} + \delta u_{ij}\}$ .

The PSG of  $\{u_{ij}^{uRVB} + \delta u_{ij}\}$  must be a subgroup of the PSG of the u-RVB state Eq.(8). In appendix C we classify all these possible PSG subgroups with the  $Z_2$  IGG, which allows us to construct all possible  $Z_2$  states around the u-RVB state. This technique was firstly developed by Wen<sup>23</sup>. We find 24 gauge inequivalent  $Z_2$  PSGs as listed in Table I in appendix C.

Can these 24  $Z_2$  SL states have a full energy gap? We find not all of them can have a gapped spinon spectrum. This can be understood starting from a Dirac dispersion of the u-RVB state. To gap out the Dirac nodes, at least one mass term in the low-energy effective theory of a given  $Z_2$  state must be allowed by symmetry. In appendix E we show that only 4 of the 24  $Z_2$  states allow mass term in the low energy theory. Thus only these 4 states are fully gapped  $Z_2$  spin liquids around u-RVB state. The other 20 states have symmetry protected gapless spinon dispersions.

These four states are state #16, #17, #19, and #22 in Table I in appendix C. We can generate their mean-field ansatzs by these PSGs. We find that up to the 3rd neighbor mean-field amplitudes  $u_{(\alpha,\beta,\gamma)}$  as shown in Fig.1, only one of these four states can be realized, which is state #19. As shown in appendix E 2, mean-field ansatzs up to the 3rd neighbor of the other three states actually have

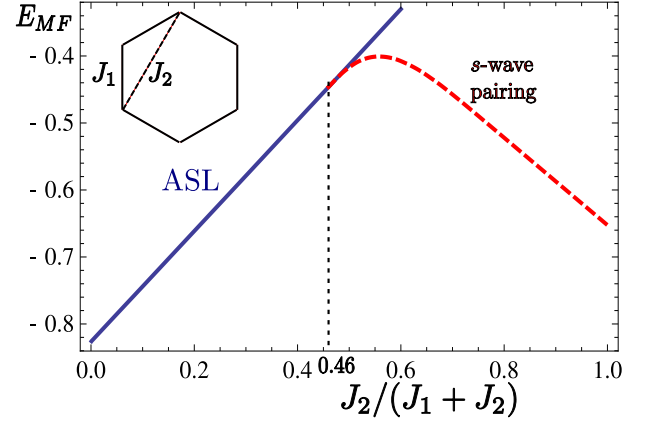


FIG. 3: Mean-field phase diagram of  $J_1 - J_2$  model by Schwinger-fermion approach.

a  $U(1)$  IGG. Only after introducing longer-range mean-field bonds can these three states have a  $Z_2$  IGG. In particular, state #16 requires 5th neighbor, state #17 requires 4th neighbor and state #22 requires 9th neighbor amplitudes, while state #19 only requires 2nd neighbor amplitudes. Because the  $t/U$  expansion of the Hubbard model give a rather short-ranged spin interaction for the SL phase found in numerics<sup>1</sup> ( $t/U \sim 1/4$ ), the other three states are unlikely to be realized in a Hubbard model on honeycomb lattice.

After choosing a proper gauge, the mean-field ansatz of #19 can be expressed as a sublattice dependent pairing of the  $f$ -spinons, as shown in Fig.1:

$$H_{MF} = \chi \sum_{\langle ij \rangle} f_{i\alpha}^\dagger f_{j\alpha} + \Delta e^{i\theta} \sum_{\langle\langle ij \rangle\rangle \in A} \epsilon_{\alpha\beta} f_{i\alpha}^\dagger f_{j\beta}^\dagger + \Delta e^{-i\theta} \sum_{\langle\langle ij \rangle\rangle \in B} \epsilon_{\alpha\beta} f_{i\alpha}^\dagger f_{j\beta}^\dagger + \text{h.c.} \quad (9)$$

and we term it as sublattice pairing state (SPS). Note that  $\theta \neq 0, \pm\pi/2, \pi$ , because otherwise the ansatz has  $U(1)$  IGG. We propose SPS to be the SL phase found in numerics.

#### IV. SCHWINGER-FERMION MEAN-FIELD STUDY OF THE $J_1 - J_2$ MODEL ON HONEYCOMB LATTICE

Can SPS be realized in the Hubbard model when  $t/U \sim 1/4$ , where numerics shows a gapped SL phase? In particular, by the Mott transition theory of Hermele<sup>27</sup>, the u-RVB (or ASL) state is in the neighborhood of the Mott transition. Can SPS be more favorable than the ASL state? To address this question, we use  $t/U$  expansion of the Hubbard model<sup>32</sup> to obtain an effective  $J_1 - J_2$  spin model on honeycomb lattice:

$$H = J_1 \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j + J_2 \sum_{\langle\langle ij \rangle\rangle} \vec{S}_i \cdot \vec{S}_j \quad (10)$$



where  $J_1$  and  $J_2$  are the 1st neighbor and 2nd neighbor antiferromagnetic coupling. Following Ref. 32, we find up to  $t^4/U^3$  order, the effective  $J_1$  and  $J_2$  are:

$$J_1 = 4t^2/U - 16t^4/U^3, \quad J_2 = 4t^4/U^3. \quad (11)$$

Naively plugging in  $t/U \sim 1/4$  gives  $J_2/J_1 \sim 1/12$ .

We use the variationally mean-field ansatz Eq.9. Note that this mean-field study is biased towards spin disordered ground state. For example, we do not include Neel order which is known to be the ground state at  $J_2 = 0$ , and we also do not include the spiral spin order which is found by semiclassical study of  $J_1 - J_2$  model<sup>33,34</sup>. The purpose of the current mean-field study is to understand whether a gapped spin liquid can be more favorable compared to the gapless ASL state when  $J_2$  is tuned up and frustration becomes important.

By minimizing the mean-field energy in Eq.(3), the phase diagram of  $J_1 - J_2$  model is obtained and shown in Fig.3, where we fix  $J_1 + J_2 = 1$  and  $E_{MF}$  is scaled from Eq.(3) by  $8/3$ . We find that when  $J_2/J_1 < 0.85$  (or  $J_2/(J_1 + J_2) < 0.46$ ), the ground state is u-RVB(or ASL) state:  $\chi \neq 0$  and  $\Delta = 0$ . When  $J_2/J_1 > 0.85$ , the ground state is an  $s$ -wave pairing state:  $\chi, \Delta \neq 0$  and  $\theta = 0$ . The  $s$ -wave pairing state opens an energy gap for spinons but has remaining  $U(1)$  gapless gauge fluctuation. Due to monopole proliferation<sup>26</sup> the  $s$ -wave pairing state is not a stable phase. In this mean-field study, the gauge fluctuations are not considered and this is the reason why we find  $s$ -wave pairing state as a ground state. Taking gauge fluctuations into account, the likely fate of the  $s$ -wave pairing state is that  $\theta$  becomes nonzero and the  $Z_2$  SPS state is realized.

We propose to study the  $J_1 - J_2$  model by Gutzwiller projected wavefunction variational approach<sup>35</sup> because it can be viewed as a method to include the gauge fluctuation. We leave this projected wavefunction study as a direction of future research, which may realize SPS as the ground state. Projected wavefunctions are also classified by PSG, so the present work also provide guideline for the search of ground states in the projected wavefunction space.

## V. DISCUSSION

In this work we completely classified the  $Z_2$  mean-field states in the Schwinger-fermion approach. Using physical argument, we identify a single state: SPS, as the possible spin liquid phase found in the recent Quantum Monte Carlo study of the Hubbard model on a honeycomb lattice<sup>1</sup>. SPS is in the neighborhood of the semimetal phase and we propose a scenario for the continuous transition connecting the two phases.

In our mean-field study of the  $J_1 - J_2$  model, the  $s$ -wave pairing state is realized for a fairly large  $J_2$ , corresponding to a fairly large  $t/U \sim 0.44$ . A higher order spin-spin effective interaction such as the 6-spin ring exchange term

and/or a more careful projected wavefunction study may realize SPS phase for a smaller  $t/U$ .

In a recent work<sup>36</sup>, Wang study the  $Z_2$  mean-field states in the Schwinger-boson approach, and identify a zero-flux SL state, which is naturally connected to a Neel ordered state by a potentially continuous phase transition. Whether the SPS found in the present work is related to Wang's result is unclear. And we leave the possible continuous transition from SPS to the Neel ordered phase as a subject of future research.

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## Appendix A: General conditions on projective symmetry groups on a honeycomb lattice

As mentioned in section II, SG on a honeycomb lattice is generated by time reversal transformation  $\mathbf{T}$ , translations along  $\vec{a}_1, \vec{a}_2$ :  $T_1, T_2$ , plaquette-centered  $60^\circ$   $C_6$  rotation, and a horizontal mirror reflection  $\sigma$  as shown in Fig.1. In the present problem, the symmetry group  $SG$  can be represented as

$$SG = \{U = \mathbf{T}^{\nu_T} \cdot T_1^{\nu_{T_1}} \cdot T_2^{\nu_{T_2}} \cdot C_6^{\nu_{C_6}} \cdot \sigma^{\nu_\sigma}\}$$

where  $\nu_{T_1}, \nu_{T_2} \in \mathbf{Z}$  and  $\nu_T, \nu_\sigma \in \mathbf{Z}_2, \nu_{C_6} \in \mathbf{Z}_6$ , since the generators satisfy

$$\mathbf{T}^2 = \sigma^2 = (C_6)^6 = 1 \quad (A1)$$

Here 1 stands for the identity element of  $SG$ . To completely determine the multiplication rule of this group, we need to identify the multiplication rule of two different generators in an order different from  $\mathbf{T}^{\nu_T} \cdot T_1^{\nu_{T_1}} \cdot T_2^{\nu_{T_2}} \cdot C_6^{\nu_{C_6}} \cdot \sigma^{\nu_\sigma}$ :

$$U\mathbf{T} = \mathbf{T}U \quad (U = T_1, T_2, C_6, \sigma) \quad (A2)$$

$$T_1 T_2 = T_2 T_1 \quad (A3)$$

$$C_6 T_1 = T_2 C_6 \quad (A4)$$

$$C_6 T_2 = T_1^{-1} T_2 C_6 \quad (A5)$$

$$\sigma T_1 = T_1 \sigma \quad (A6)$$

$$\sigma T_2 = T_1 T_2^{-1} \sigma \quad (A7)$$

$$\sigma C_6 = C_6^{-1} \sigma \quad (A8)$$

The above relations can be written in an alternative way

$$\mathbf{T}^2 = \sigma^2 = (C_6)^6 = 1 \quad (A9)$$

$$\mathbf{T}U\mathbf{T}^{-1}U^{-1} = 1 \quad (U = T_1, T_2, C_6, \sigma) \quad (A10)$$

$$T_1 T_2 T_1^{-1} T_2^{-1} = 1 \quad (A11)$$

$$T_2^{-1} C_6 T_1 C_6^{-1} = 1 \quad (A12)$$

$$T_1^{-1} C_6 T_1 T_2^{-1} C_6^{-1} = 1 \quad (A13)$$

$$T_1^{-1} \sigma T_1 \sigma^{-1} = 1 \quad (A14)$$

$$T_2^{-1} \sigma T_1 T_2^{-1} \sigma^{-1} = 1 \quad (A15)$$

$$\sigma C_6 \sigma C_6 = 1 \quad (A16)$$

which determines the inverse of all the group elements.

As introduced in section II, the mean-field ansatz  $\{u_{ij}\}$  of a spin liquid is invariant under the action of any element  $G_U U$  of a projective symmetry group (PSG). The multiplication rule of the symmetry group would immediately enforce the following constraints on a PSG by its definition: if  $U_1 U_2 = U_3$  then

$$\begin{aligned} G_{U_1} U_1 G_{U_2} U_2 (\{u_{ij}\}) &= G_{U_3} U_3 (\{u_{ij}\}) \implies \\ [G_{U_1} (U_1 U_2(i)) G_{U_2} (U_2(i))] u_{ij} [G_{U_1} (U_1 U_2(i)) G_{U_2} (U_2(i))]^\dagger & \\ = [G_{U_3} (U_3(i))] u_{ij} [G_{U_3} (U_3(i))]^\dagger, \quad \forall i, j & \quad (A17) \end{aligned}$$

On the other hand, we know those pure gauge transformations, under which the mean-field ansatz  $\{u_{ij}\}$  is invariant, constitute a subgroup of PSG, coined the invariant gauge group (IGG):

$$IGG = \{W_i | W_i u_{ij} W_j^\dagger = u_{ij}, \quad W_i \in SU(2)\} \quad (A18)$$

Therefore from (A17) we have the following constraints on the elements of a PSG

$$[G_{U_1 U_2} (U_1 U_2(i))]^\dagger G_{U_1} (U_1 U_2(i)) G_{U_2} (U_2(i)) = \mathcal{G} \in IGG$$

The above condition holds for any two group elements  $U_1, U_2$  of SG. Similar with SG, we can choose a set of generators in any given PSG:  $\{G_{T_1} T_1, G_{T_2} T_2, G_{\mathbf{T}} \mathbf{T}, G_{C_6} C_6, G_{\sigma} \sigma\}$ . Any given element in PSG can be written in the standard form:

$$\begin{aligned} G_U U &= (G_{\mathbf{T}} \mathbf{T})^{\nu_{\mathbf{T}}} \cdot (G_{T_1} T_1)^{\nu_{T_1}} \cdot (G_{T_2} T_2)^{\nu_{T_2}} \\ &\cdot (G_{C_6} C_6)^{\nu_{C_6}} \cdot (G_{\sigma} \sigma)^{\nu_{\sigma}} \end{aligned} \quad (A19)$$

Since the multiplication rule of SG on a honeycomb lattice is completely determined by (A1)-(A8), or equivalently (A9)-(A16), the only independent constraints on the PSG generators are the following:

$$\begin{aligned} (G_{\mathbf{T}} \mathbf{T})^2 &\in IGG & (A20) \\ (G_{\sigma} \sigma)^2 &\in IGG \\ (G_{C_6} C_6)^6 &\in IGG \\ (G_{T_1} T_1)^{-1} (G_{T_2} T_2)^{-1} (G_{T_1} T_1) (G_{T_2} T_2) &\in IGG \\ (G_{T_1} T_1)^{-1} (G_{\mathbf{T}} \mathbf{T})^{-1} (G_{T_1} T_1) (G_{\mathbf{T}} \mathbf{T}) &\in IGG \\ (G_{T_2} T_2)^{-1} (G_{\mathbf{T}} \mathbf{T})^{-1} (G_{T_2} T_2) (G_{\mathbf{T}} \mathbf{T}) &\in IGG \\ (G_{T_2} T_2)^{-1} (G_{C_6} C_6) (G_{T_1} T_1) (G_{C_6} C_6)^{-1} &\in IGG \\ (G_{T_1} T_1)^{-1} (G_{C_6} C_6) (G_{T_1} T_1) (G_{T_2} T_2)^{-1} (G_{C_6} C_6)^{-1} &\in IGG \\ (G_{\mathbf{T}} \mathbf{T})^{-1} (G_{C_6} C_6)^{-1} (G_{\mathbf{T}} \mathbf{T}) (G_{C_6} C_6) &\in IGG \\ (G_{T_1} T_1)^{-1} (G_{\sigma} \sigma) (G_{T_1} T_1) (G_{\sigma} \sigma)^{-1} &\in IGG \\ (G_{T_2} T_2)^{-1} (G_{\sigma} \sigma) (G_{T_2} T_2)^{-1} (G_{\sigma} \sigma)^{-1} &\in IGG \\ (G_{\sigma} \sigma) (G_{C_6} C_6) (G_{\sigma} \sigma) (G_{C_6} C_6) &\in IGG \\ (G_{\mathbf{T}} \mathbf{T})^{-1} (G_{\sigma} \sigma)^{-1} (G_{\mathbf{T}} \mathbf{T}) (G_{\sigma} \sigma) &\in IGG \end{aligned}$$

or more specifically

$$\begin{aligned} [G_{\mathbf{T}}(i)]^2 &\in IGG, & (A21) \\ G_{\sigma}(\sigma(i)) G_{\sigma}(i) &\in IGG, \\ G_{C_6}(C_6^{-1}(i)) G_{C_6}(C_6^{-2}(i)) G_{C_6}(C_6^3(i)) \\ \cdot G_{C_6}(C_6^2(i)) G_{C_6}(C_6(i)) G_{C_6}(i) &\in IGG, \\ G_{T_1}^{-1}(T_2^{-1} T_1(i)) G_{T_2}^{-1}(T_1(i)) G_{T_1}(T_1(i)) G_{T_2}(i) &\in IGG, \\ G_{T_1}^{-1}(T_1(i)) G_{\mathbf{T}}^{-1}(T_1(i)) G_{T_1}(T_1(i)) G_{\mathbf{T}}(i) &\in IGG, \\ G_{T_2}^{-1}(T_2(i)) G_{\mathbf{T}}^{-1}(T_2(i)) G_{T_2}(T_2(i)) G_{\mathbf{T}}(i) &\in IGG, \\ G_{T_2}^{-1}(T_2(i)) G_{C_6}(T_2(i)) G_{T_1}(T_1 C_6^{-1}(i)) G_{C_6}^{-1}(i) &\in IGG, \\ G_{T_1}^{-1}(T_1(i)) G_{C_6}(T_1(i)) G_{T_1}(C_6^{-1} T_1(i)) \\ \cdot G_{T_2}^{-1}(C_6^{-1}(i)) G_{C_6}^{-1}(i) &\in IGG, \\ G_{\mathbf{T}}^{-1}(C_6^{-1}(i)) G_{C_6}^{-1}(i) G_{\mathbf{T}}(i) G_{C_6}(i) &\in IGG, \\ G_{T_1}^{-1}(T_1(i)) G_{\sigma}(T_1(i)) G_{T_1}(T_1 \sigma^{-1}(i)) G_{\sigma}^{-1}(i) &\in IGG, \\ G_{T_2}^{-1}(T_2(i)) G_{\sigma}(T_2(i)) G_{T_1}(\sigma T_2(i)) G_{T_2}^{-1}(\sigma(i)) G_{\sigma}^{-1}(i) &\in IGG, \\ G_{\sigma}(i) G_{C_6}(\sigma(i)) G_{\sigma}(\sigma C_6(i)) G_{C_6}(C_6(i)) &\in IGG, \\ G_{\mathbf{T}}^{-1}(\sigma(i)) G_{\sigma}^{-1}(i) G_{\mathbf{T}}(i) G_{\sigma}(i) &\in IGG. \end{aligned}$$

Above are all the general consistent conditions to be satisfied by the generators of a PSG on a honeycomb lattice.

We will use  $(x_1, x_2, s)$  to label a site  $i$  in a honeycomb lattice, where  $x_1, x_2$  are the coordinates of the unit cell in basis  $\vec{a}_1, \vec{a}_2$  and  $s = 0, 1$  for  $A$  and  $B$  sublattice respectively. For convenience, we summarize the coordinate transformation of all the generators in the symmetry group on a honeycomb lattice as follows:

$$\begin{aligned} \mathbf{T} : \quad (x_1, x_2, s) &\rightarrow (x_1, x_2, s), & (A22) \\ T_1 : \quad (x_1, x_2, s) &\rightarrow (x_1 + 1, x_2, s), \\ T_2 : \quad (x_1, x_2, s) &\rightarrow (x_1, x_2 + 1, s), \\ \sigma : \quad (x_1, x_2, s) &\rightarrow (x_1 + x_2, -x_2, 1 - s), \\ C_6 : \quad (x_1, x_2, 0) &\rightarrow (1 - x_2, x_1 + y_1 - 1, 1) \\ &\quad (x_1, x_2, 1) \rightarrow (-x_2, x_1 + y_1, 0) \end{aligned}$$

## Appendix B: Classification of $Z_2$ projective symmetry groups on a honeycomb lattice

As discussed in section II, the problem of classifying all possible  $Z_2$  Schwinger-fermion mean-field states is mathematically reduced to finding all possible PSGs. Let us firstly find all algebraic PSGs.

### 1. General discussions

In the case of  $Z_2$  spin liquids, the IGG of the corresponding PSG is a  $Z_2$  group:  $IGG = \{\pm \tau^0\}$ . The

constraints listed in appendix A now becomes

$$[G_{\mathbf{T}}(i)]^2 = \eta_{\mathbf{T}} \tau^0, \quad (\text{B1})$$

$$G_{\sigma}(\sigma(i))G_{\sigma}(i) = \eta_{\sigma} \tau^0, \quad (\text{B2})$$

$$G_{C_6}(C_6^{-1}(i))G_{C_6}(C_6^{-2}(i))G_{C_6}(C_6^3(i)) \quad (\text{B3})$$

$$\cdot G_{C_6}(C_6^2(i))G_{C_6}(C_6(i))G_{C_6}(i) = \eta_{C_6} \tau^0, \quad (\text{B4})$$

$$G_{T_1}^{-1}(T_2^{-1}T_1(i))G_{T_2}^{-1}(T_1(i)) \cdot G_{T_1}(T_1(i))G_{T_2}(i) = \eta_{12} \tau^0, \quad (\text{B5})$$

$$G_{T_1}^{-1}(T_1(i))G_{\mathbf{T}}^{-1}(T_1(i))G_{T_1}(T_1(i))G_{\mathbf{T}}(i) = \eta_{1\mathbf{T}} \tau^0 \quad (\text{B6})$$

$$G_{T_2}^{-1}(T_2(i))G_{\mathbf{T}}^{-1}(T_2(i))G_{T_2}(T_2(i))G_{\mathbf{T}}(i) = \eta_{2\mathbf{T}} \tau^0 \quad (\text{B7})$$

$$G_{T_2}^{-1}(T_2(i))G_{C_6}(T_2(i)) \cdot G_{T_1}(T_1C_6^{-1}(i))G_{C_6}^{-1}(i) = \eta_{C_61} \tau^0, \quad (\text{B8})$$

$$G_{T_1}^{-1}(T_1(i))G_{C_6}(T_1(i))G_{T_1}(C_6^{-1}T_1(i)) \cdot G_{T_2}^{-1}(C_6^{-1}(i))G_{C_6}^{-1}(i) = \eta_{C_62} \tau^0, \quad (\text{B9})$$

$$G_{\mathbf{T}}^{-1}(C_6^{-1}(i))G_{C_6}^{-1}(i)G_{\mathbf{T}}(i)G_{C_6}(i) = \eta_{C_6\mathbf{T}} \tau^0, \quad (\text{B10})$$

$$G_{T_1}^{-1}(T_1(i))G_{\sigma}(T_1(i)) \cdot G_{T_1}(T_1\sigma^{-1}(i))G_{\sigma}^{-1}(i) = \eta_{\sigma1} \tau^0, \quad (\text{B11})$$

$$G_{T_2}^{-1}(T_2(i))G_{\sigma}(T_2(i))G_{T_1}(\sigma T_2(i)) \cdot G_{T_2}^{-1}(\sigma(i))G_{\sigma}^{-1}(i) = \eta_{\sigma2} \tau^0, \quad (\text{B12})$$

$$G_{\sigma}(i)G_{C_6}(\sigma(i)) \cdot G_{\sigma}(\sigma C_6(i))G_{C_6}(C_6(i)) = \eta_{\sigma C_6} \tau^0, \quad (\text{B13})$$

$$G_{\mathbf{T}}^{-1}(\sigma(i))G_{\sigma}^{-1}(i)G_{\mathbf{T}}(i)G_{\sigma}(i) = \eta_{\sigma\mathbf{T}} \tau^0. \quad (\text{B14})$$

where all the  $\eta$ 's take value of  $\pm 1$ . Not all of these conditions are gauge independent. Because we can re-choose the gauge part of generators such as  $G_{T_1}, G_{T_2} \dots$  by multiplying them by  $-\tau^0$  (an element of IGG), only those conditions in which the same generator shows up twice are gauge independent. We can use this gauge dependence to simplify these conditions. Because  $G_{T_1}(G_{T_2})$  only show up once in the equation of  $\eta_{C_61}(\eta_{C_62})$ , we can always choose a gauge such that  $\eta_{C_61} = \eta_{C_62} = 1$ . All other  $\eta$ 's are gauge independent.

In the following we will determine all the possible PSG's with different (gauge inequivalent) elements  $\{G_U(i)\}$ . These different PSG's characterize all the different type of  $Z_2$  spin liquids on a honeycomb lattice, which might be constructed from mean-field ansatz  $\{u_{ij}\}$ .

First notice that under a local  $SU(2)$  gauge transformation  $u_{ij} \rightarrow W_i u_{ij} W_j^\dagger$ , the PSG elements transform as  $G_U(i) \rightarrow W_i G_U(i) W_{U^{-1}(i)}^\dagger$ . Making use of such a degree of freedom, we can always choose proper gauge so that

$$G_{T_1}(x_1, x_2, s) = G_{T_2}(0, x_2, s) = \tau^0, \quad x_1, x_2 \in \mathbb{Z}.$$

Now taking (B5) into account, we have  $G_{T_2}(\{x_1 + 1, x_2, s\}) = \eta_{12} G_{T_2}(\{x_1, x_2, s\})$  and therefore

$$G_{T_1}(x_1, x_2, s) = \tau^0 \quad (\text{B15})$$

$$G_{T_2}(x_1, x_2, s) = \eta_{12}^x \tau^0$$

Meanwhile, from (B1), (B6) and (B7) we can immediately see that  $\eta_{1\mathbf{T}} = \eta_{2\mathbf{T}} = 1$ , and the gauge inequivalent choices of  $G_{\mathbf{T}}(i)$  are the following

$$G_{\mathbf{T}}(x_1, x_2, s) = g_{\mathbf{T}}(s) = \begin{cases} \eta_t^s \tau^0, & \eta_{\mathbf{T}} = 1 \\ i\tau^3, & \eta_{\mathbf{T}} = -1 \end{cases} \quad (\text{B16})$$

where  $\eta_t = \pm 1$ .

As discussed earlier, we can always choose a proper gauge so that  $\eta_{C_61} = \eta_{C_62} = 1$ . Then from conditions (B8) and (B9) we see that

$$G_{C_6}(x_1, x_2, s) = \eta_{12}^{x_1 x_2 + x_1(x_1-1)/2} g_{C_6}(s) \quad (\text{B17})$$

similarly from conditions (B11) and (B12) we have

$$G_{\sigma}(x_1, x_2, s) = \eta_{\sigma 1}^{x_1} \eta_{\sigma 2}^{x_2} \eta_{12}^{x_2(x_2-1)/2} g_{\sigma}(s) \quad (\text{B18})$$

where  $g_{C_6}(s), g_{\sigma}(s) \in SU(2)$ . Note that (B2) and (B13) give further constraints to the above expression (B18):

$$\eta_{\sigma 1} = \eta_{\sigma 2} = \eta_{12} \quad (\text{B19})$$

Now we see the elements of PSG can be expressed as

$$G_{T_1}(x_1, x_2, s) = \tau^0 \quad (\text{B20})$$

$$G_{T_2}(x_1, x_2, s) = \eta_{12}^{x_1} \tau^0$$

$$G_{\mathbf{T}}(x_1, x_2, s) = g_{\mathbf{T}}(s) \quad (\text{B21})$$

$$G_{C_6}(x_1, x_2, s) = \eta_{12}^{x_1 x_2 + x_1(x_1-1)/2} g_{C_6}(s)$$

$$G_{\sigma}(x_1, x_2, s) = \eta_{12}^{x_1 + x_2(x_2+1)/2} g_{\sigma}(s)$$

Consistent conditions (B2), (B4), (B10), (B13) and (B14) correspond to the following constraints on  $SU(2)$  matrices  $g_{C_6}(s), g_{\sigma}(s)$ :

$$g_{\sigma}(0)g_{\sigma}(1) = \eta_{\sigma} \tau^0, \quad (\text{B22})$$

$$[g_{C_6}(s)g_{C_6}(1-s)]^3 = \eta_{C_6} \eta_{12} \tau^0,$$

$$g_{\mathbf{T}}(s)g_{C_6}(s) = g_{C_6}(s)g_{\mathbf{T}}(1-s)\eta_{C_6\mathbf{T}}$$

$$g_{\mathbf{T}}(s)g_{\sigma}(s) = g_{\sigma}(s)g_{\mathbf{T}}(1-s)\eta_{\sigma\mathbf{T}}$$

$$g_{\sigma}(s)g_{C_6}(1-s) = \begin{cases} \lambda_{C_6}^s \tau^0, & \eta_{\sigma C_6} = 1 \\ i\hat{n}_s \cdot \vec{\tau}, & \eta_{\sigma C_6} = -1 \end{cases}$$

where  $\lambda_{C_6} = \pm 1$  and  $\hat{n}_s$  is a unit vector.

## 2. A summary of 160 different PSG's

Below we summarize all the 160 possible PSG's obtained through solving (B22). We use capital Roman numerals (I) and (II) to label  $g_{\mathbf{T}} = \eta_t^s \tau^0$  and  $g_{\mathbf{T}} = i\tau^3$  respectively. Roman numerals (i) and (ii) are used to label  $\eta_{C_6\mathbf{T}} = \pm 1$  respectively. (A) and (B) are used to label  $\eta_{\sigma C_6} = \pm 1$  respectively. Finally ( $\alpha$ ) and ( $\beta$ ) are used to label  $\eta_{\sigma\mathbf{T}}$  respectively.

$$(I) \quad g_{\sigma\mathbf{T}} = \eta_t^s \tau^0:$$

It's easy to see that  $\eta_{C_6 T} = \eta_{\sigma T} = \eta_t$  from (B22), so there is the only possibility among (i) and (ii).

- (A)  $g_{\sigma}(s) = \lambda_{C_6}^s g_{C_6}^{-1}(1-s)$ :  
we have  $\lambda_{C_6} = \eta_{\sigma} \eta_{C_6} \eta_{12}$  and

$$\begin{aligned} g_{C_6}(0) &= \tau^0, \\ g_{C_6}(1) &= g_{\sigma}(0) = \eta_{C_6} \eta_{12} \tau^0, \\ g_{\sigma}(1) &= \eta_{\sigma} \eta_{C_6} \eta_{12} \tau^0. \end{aligned} \quad (\text{B23})$$

This represents  $2^4 = 16$  different PSG's in the class (I)(A) since  $\eta_t, \eta_{C_6}, \eta_{\sigma}, \eta_{12} = \pm 1$ .

- (B)  $g_{\sigma}(s) g_{C_6}(1-s) = i \hat{n}_s \cdot \vec{\tau}$ :

Choosing a proper gauge (so that  $g_{C_6}(0) = \tau^0$ ) we have

$$\begin{aligned} g_{C_6}(0) &= \tau^0, \\ g_{C_6}(1) &= \eta_{C_6} \eta_{12} e^{i\psi_3 \tau^3}, \\ g_{\sigma}(0) &= i \tau^1 \eta_{C_6} \eta_{12} e^{-i\psi_3 \tau^3}, \\ g_{\sigma}(1) &= -i \eta_{\sigma} \eta_{C_6} \eta_{12} e^{i\psi_3 \tau^3} \tau^1. \end{aligned} \quad (\text{B24})$$

where  $\psi_3 \equiv 0, \pm 2\pi/3$  stand for the multiples of  $2\pi/3$  mod  $2\pi$ . There are  $2^4 \times 3 = 48$  different PSG's in this class (I)(B).

- (II)  $g_T(s) = i \tau^3$ :

- (i)  $\eta_{C_6 T} = 1$ :

- (A)  $g_{\sigma}(s) = \lambda_{C_6}^s g_{C_6}^{-1}(1-s)$ :

in this case  $\lambda_{C_6} = \eta_{\sigma} \eta_{C_6} \eta_{12}$ , so we have

$$\begin{aligned} g_{C_6}(0) &= \tau^0, \\ g_{\sigma}(0) &= g_{C_6}(1) = \eta_{C_6} \eta_{12} \tau^0, \\ g_{\sigma}(1) &= \eta_{\sigma} \eta_{C_6} \eta_{12} \tau^0. \end{aligned} \quad (\text{B25})$$

there are  $2^3 = 8$  different PSG's in the class (II)(i)(A).

- (B)  $g_{\sigma}(s) g_{C_6}(1-s) = i \hat{n}_s \cdot \vec{\tau}$ :

- (\alpha)  $\eta_{\sigma T} = 1$ , i.e.  $[g_{\sigma}(s), \tau^3] = 0$ :

here we have

$$\begin{aligned} g_{C_6}(0) &= \tau^0, \\ g_{C_6}(1) &= \eta_{C_6} \eta_{12} \tau^0, \\ g_{\sigma}(0) &= -i \eta_{\sigma} \tau^3, \\ g_{\sigma}(1) &= i \tau^3. \end{aligned} \quad (\text{B26})$$

there are  $2^3 = 8$  different PSG's in the class (II)(i)(B)(\alpha).

- (\beta)  $\eta_{\sigma T} = -1$ , i.e.  $\{g_{\sigma}(s), \tau^3\} = 0$ :

here we have

$$\begin{aligned} g_{C_6}(0) &= \tau^0, \\ g_{C_6}(1) &= \eta_{C_6} \eta_{12} e^{i\psi_3 \tau^3}, \\ g_{\sigma}(0) &= -i \eta_{\sigma} \tau^1, \\ g_{\sigma}(1) &= i \tau^1. \end{aligned} \quad (\text{B27})$$

there are  $2^3 \times 3 = 24$  different PSG's in the class (II)(i)(B)(\beta) since  $\psi_3 = 0, \pm 2\pi/3$ .

- (ii)  $\eta_{C_6 T} = -1$ :

- (A)  $g_{\sigma}(s) = \lambda_{C_6}^s g_{C_6}^{-1}(1-s)$ :

here we must have  $\eta_{\sigma T} = -1$ ,  $\lambda_{C_6} = \eta_{\sigma} \eta_{C_6} \eta_{12}$  and

$$\begin{aligned} g_{C_6}(0) &= i \tau^1, \\ g_{C_6}(1) &= -i \eta_{C_6} \eta_{12} \tau^1, \\ g_{\sigma}(0) &= i \eta_{C_6} \eta_{12} \tau^1, \\ g_{\sigma}(1) &= -i \eta_{\sigma} \eta_{C_6} \eta_{12} \tau^1. \end{aligned} \quad (\text{B28})$$

there are  $2^3 = 8$  different PSG's in the class (II)(ii)(A).

- (B)  $g_{\sigma}(s) g_{C_6}(1-s) = i \hat{n}_s \cdot \vec{\tau}$ :

- (\alpha)  $\eta_{\sigma T} = 1$ , i.e.  $[g_{\sigma}(s), \tau^3] = 0$ :

here we have

$$\begin{aligned} g_{C_6}(0) &= i \tau^1, \\ g_{C_6}(1) &= -i \eta_{C_6} \eta_{12} \tau^1 e^{i\psi_3 \tau^3}, \\ g_{\sigma}(0) &= \tau^0, \\ g_{\sigma}(1) &= \eta_{\sigma} \tau^0. \end{aligned} \quad (\text{B29})$$

there are  $2^3 \times 3 = 24$  different PSG's in the class (II)(ii)(B)(\alpha) since  $\psi_3 = 0, \pm 2\pi/3$ .

- (\beta)  $\eta_{\sigma T} = -1$ , i.e.  $\{g_{\sigma}(s), \tau^3\} = 0$ :

here we have

$$\begin{aligned} g_{C_6}(0) &= i \tau^1, \\ g_{C_6}(1) &= -i \eta_{C_6} \eta_{12} \tau^1 e^{i\psi_3 \tau^3}, \\ g_{\sigma}(0) &= i \tau^1, \\ g_{\sigma}(1) &= -i \eta_{\sigma} \tau^1. \end{aligned} \quad (\text{B30})$$

there are  $2^3 \times 3 = 24$  different PSG's in the class (II)(ii)(B)(\beta) since  $\psi_3 = 0, \pm 2\pi/3$ .

To summarize, above are the 160 different (algebraic) PSG's with  $IGG = \{\pm \tau^0\}$  on a honeycomb lattice. They represent different  $Z_2$  spin liquid states on a honeycomb lattice, which possess all the symmetries of the honeycomb lattice generated by  $\{\mathbf{T}, T_1, T_2, \sigma, C_6\}$ . We also want to emphasize that any solution to the set of equation (B1)-(B14) may look different, but it will be gauge equivalent to one of these 160 PSG's.

On the other hand, such a (algebraic) PSG really corresponds to a spin liquid if and only if it can be realized by a mean-field ansatz  $\{u_{ij}\}$  on a honeycomb lattice<sup>23</sup>. In fact, not all of these algebraic PSGs can be realized by an ansatz. After the time-reversal transformation, the mean field amplitude changes sign<sup>23</sup>:  $\mathbf{T}(u_{ij}) = -u_{ij}$ . Gauge transformation  $G_{\mathbf{T}}$  must change the sign again:

$$-u_{ij} = G_{\mathbf{T}}(i) u_{ij} G_{\mathbf{T}}(j)^{\dagger} \quad (\text{B31})$$

If in an algebraic PSG,  $G_{\mathbf{T}}(i) = \tau^0$  independent of site,  $u_{ij}$  must vanish.

Clearly at least 32 algebraic PSG's among the total 160 types cannot be realized by any mean-field ansatz  $\{u_{ij}\}$ . These are the PSG's with  $G_{\mathbf{T}}(i) = g_{\mathbf{T}}(s) = \tau^0$  in the class (I)(I)(A)&(B). Since under time reversion  $\mathbf{T}$  we require  $-u_{ij} = G_{\mathbf{T}}(i) u_{ij} G_{\mathbf{T}}^{\dagger}(j) = u_{ij}$ , this leads to the vanishing of all bonds  $\{u_{ij} \equiv 0\}$  in the mean-field ansatz. Therefore, there are at the most 128 possible  $Z_2$  spin liquids that can be realized by a mean-field ansatz on a honeycomb lattice.



### Appendix C: Classification of $Z_2$ projective symmetry groups around u-RVB ansatz

In this section we focus on those  $Z_2$  spin liquids near the u-RVB state, which is discussed in section III. These  $Z_2$  spin liquids are plausibly connected to a semimetal through a continuous phase transition. The u-RVB state is realized by the following ansatz:

$$u_{ij} = (-1)^{s_i} i \chi \tau^0 \quad (C1)$$

its mean-field bond is only nonzero between nearest neighbors  $\langle ij \rangle$ , which have different sublattice indices  $s_i = 1 - s_j$ . By definition, its PSG has the following form:

$$\begin{aligned} G_{T_1}(x_1, x_2, s) &= g_1, \\ G_{T_2}(x_1, x_2, s) &= g_2, \\ G_{\mathbf{T}}(x_1, x_2, s) &= (-1)^s g_{\mathbf{T}}, \\ G_{C_6}(x_1, x_2, s) &= (-1)^s g_{C_6}, \\ G_{\sigma}(x_1, x_2, s) &= (-1)^s g_{\sigma}. \end{aligned} \quad (C2)$$

where  $g_1, g_2, g_{\mathbf{T}}, g_{C_6}, g_{\sigma} \in SU(2)$ . To find out those  $Z_2$  spin liquids around such a u-RVB state, we need to trace those PSG's with  $IGG = \{\pm\tau^0\}$  that looks like (C2). Consistent conditions (B1)-(B14) now corresponds to constraints on the  $SU(2)$  matrices  $\{g_1, g_2, g_{\mathbf{T}}, g_{C_6}, g_{\sigma}\}$ :

$$\begin{aligned} g_1^{-1} g_2^{-1} g_1 g_2 &= \xi_{12} \tau^0, & g_{\mathbf{T}}^2 &= \xi_{\mathbf{T}} \tau^0, \\ g_1^{-1} g_{\mathbf{T}}^{-1} g_1 g_{\mathbf{T}} &= \xi_{1\mathbf{T}} \tau^0, & g_2^{-1} g_{\mathbf{T}}^{-1} g_2 g_{\mathbf{T}} &= \xi_{2\mathbf{T}} \tau^0, \\ g_2^{-1} g_{C_6} g_1 g_{C_6}^{-1} &= \xi_{C_6 1} \tau^0, & g_1^{-1} g_{C_6} g_2 g_{C_6}^{-1} &= \xi_{C_6 2} \tau^0, \\ g_{\mathbf{T}}^{-1} g_{C_6}^{-1} g_{\mathbf{T}} g_{C_6} &= \xi_{C_6 \mathbf{T}} \tau^0, & g_{C_6}^6 &= \xi_{C_6} \tau^0, \\ g_1^{-1} g_{\sigma} g_1 g_{\sigma}^{-1} &= \xi_{\sigma 1} \tau^0, & g_2^{-1} g_{\sigma} g_2 g_{\sigma}^{-1} &= \xi_{\sigma 2} \tau^0, \\ g_{\sigma} g_{C_6} g_{\sigma} g_{C_6} &= \xi_{\sigma C_6} \tau^0, & g_{\mathbf{T}}^{-1} g_{\sigma}^{-1} g_{\mathbf{T}} g_{\sigma} &= \xi_{\sigma \mathbf{T}} \tau^0, \\ g_{\sigma}^2 &= \xi_{\sigma} \tau^0. \end{aligned} \quad (C3)$$

where all  $\xi$ 's take value of  $\pm 1$ . Again, as discussed in appendix B we can always make  $\xi_{C_6 1} = \xi_{C_6 2} = 1$  by choosing a proper gauge. After solving eqs. (C3), we find out there are 24 gauge inequivalent solutions in total, as summarized in TABLE I. In other words, there are 24 different  $Z_2$  spin liquid around the u-RVB state, suggesting that they are promising candidates of the spin liquid connected to a semimetal on honeycomb lattice through a continuous phase transition.

### Appendix D: Consistent conditions on the mean-field ansatz $\{u_{ij}\}$ on a honeycomb lattice

In this section we derive the consistent conditions on an arbitrary mean-field bond  $u_{ij}$ , which realizes a spin liquid with a certain PSG on a honeycomb lattice. The basic idea is to find all possible symmetry group elements that transform the two lattice sites  $\{i, j\}$  into itself  $\{i, j\}$  or into each other  $\{j, i\}$ , so that the corresponding PSG

#	$g_{\mathbf{T}}$	$g_{\sigma}$	$g_{C_6}$	$g_1$	$g_2$
1	$\tau^0$	$\tau^0$	$\tau^0$	$\tau^0$	$\tau^0$
2	$\tau^0$	$\tau^0$	$i\tau^3$	$\tau^0$	$\tau^0$
3	$\tau^0$	$\tau^0$	$i\tau^3$	$e^{i2\pi/3\tau^1}$	$e^{-i2\pi/3\tau^1}$
4	$\tau^0$	$i\tau^3$	$i\tau^3$	$\tau^0$	$\tau^0$
5	$\tau^0$	$i\tau^3$	$i\tau^3$	$\tau^0$	$\tau^0$
6	$\tau^0$	$i\tau^3$	$i\tau^1$	$\tau^0$	$\tau^0$
7	$\tau^0$	$i\tau^3$	$e^{i\pi/6\tau^1}$	$\tau^0$	$\tau^0$
8	$\tau^0$	$i\tau^3$	$e^{i\pi/3\tau^1}$	$\tau^0$	$\tau^0$
9	$\tau^0$	$i\tau^3$	$i\tau^1$	$e^{i2\pi/3\tau^3}$	$e^{-i2\pi/3\tau^3}$
10	$\tau^0$	$i\tau^3$	$e^{i2\pi/3\tau^1}$	$i(\frac{\tau^1}{\sqrt{3}} - \sqrt{\frac{2}{3}}\tau^2)$	$i(\frac{\tau^3}{\sqrt{2}} - \frac{\tau^2}{\sqrt{6}} - \frac{\tau^1}{\sqrt{3}})$
11	$i\tau^3$	$\tau^0$	$\tau^0$	$\tau^0$	$\tau^0$
12	$i\tau^3$	$\tau^0$	$i\tau^3$	$\tau^0$	$\tau^0$
13	$i\tau^3$	$\tau^0$	$i\tau^1$	$\tau^0$	$\tau^0$
14	$i\tau^3$	$\tau^0$	$i\tau^1$	$e^{i2\pi/3\tau^3}$	$e^{-i2\pi/3\tau^3}$
15	$i\tau^3$	$i\tau^3$	$\tau^0$	$\tau^0$	$\tau^0$
16	$i\tau^3$	$i\tau^3$	$i\tau^3$	$\tau^0$	$\tau^0$
17	$i\tau^3$	$i\tau^3$	$i\tau^1$	$\tau^0$	$\tau^0$
18	$i\tau^3$	$i\tau^3$	$i\tau^1$	$e^{i2\pi/3\tau^3}$	$e^{-i2\pi/3\tau^3}$
19	$i\tau^3$	$i\tau^1$	$i\tau^1$	$\tau^0$	$\tau^0$
20	$i\tau^3$	$i\tau^1$	$i\tau^2$	$\tau^0$	$\tau^0$
21	$i\tau^3$	$i\tau^1$	$\tau^0$	$\tau^0$	$\tau^0$
22	$i\tau^3$	$i\tau^1$	$i\tau^3$	$\tau^0$	$\tau^0$
23	$i\tau^3$	$i\tau^1$	$e^{i\pi/6\tau^3}$	$\tau^0$	$\tau^0$
24	$i\tau^3$	$i\tau^1$	$e^{i\pi/3\tau^3}$	$\tau^0$	$\tau^0$

TABLE I: A summary of all 24 different PSG's with  $IGG = \{\pm\tau^0\}$  around the u-RVB ansatz. They correspond to 24 different  $Z_2$  spin liquids near the u-RVB state.

elements must transform mean-field bond  $u_{ij}$  into itself  $u_{ij}$  or its Hermitian conjugate  $u_{ij}^\dagger = u_{ji}$ .

As a special case, the identity element 1 always transform a bond into itself: correspondingly in PSG the  $IGG$  elements (e.g.  $\tau^0$  for a  $Z_2$  ansatz) always transform any bond  $u_{ij}$  into itself. This is nothing but the definition of invariant gauge group (IGG).

Now we need to look at nontrivial symmetry group elements which transform two lattice sites (connected by the bond) into itself or into each other. Without loss of generality, we consider the following bond

$$\langle x_1, x_2, s \rangle \equiv u_{(x_1, x_2, s)(0, 0, 0)} \quad (D1)$$

#### 1. Regarding time reversal $\mathbf{T}$

Any element of the symmetry group can be written as

$$U = \mathbf{T}^{\nu_{\mathbf{T}}} \cdot T_1^{\nu_{T_1}} \cdot T_2^{\nu_{T_2}} \cdot \sigma^{\nu_{\sigma}} \cdot C_6^{\nu_{C_6}} \quad (D2)$$

First we study the consistent conditions from time reversal transformation  $\mathbf{T}$  and then turn to other group elements.

Notice that time reversal  $\mathbf{T}$  doesn't change anything except the sign of bond:

$$G_{\mathbf{T}}(i)u_{ij}[G_{\mathbf{T}}(j)]^\dagger = -u_{ij} \quad (D3)$$

so this bond must satisfy the following constraint:

$$\begin{aligned} G_{\mathbf{T}}(x_1, x_2, s) \langle x_1, x_2, s \rangle \\ = -\langle x_1, x_2, s \rangle G_{\mathbf{T}}(0, 0, 0) \end{aligned} \quad (\text{D4})$$

## 2. Conditions on a bond within the same sublattice: $s = 0$

First we study  $s = 0$  case, *i.e.* a bond within the same sublattice. Since both mirror reflection  $\sigma$  and  $\pi/3$  rotation  $C_6$  will change the sublattice index  $s$  while the translations  $T_1, T_2$  don't, we must have an even number of reflection and rotation, *i.e.*  $\nu_\sigma + \nu_{C_6} = 0 \pmod 2$  to transform the bond to itself (or its Hermitian conjugate).

From (A22) it's easy to check the 5 nontrivial elements consisting of  $\{\sigma, C_6\}$ :

$$\begin{aligned} C_6^2(x_1, x_2, 0) &= (1 - x_1 - x_2, x_1, 0), \\ C_6^{-2}(x_1, x_2, 0) &= (x_2, 1 - x_1 - x_2, 0), \\ \sigma C_6(x_1, x_2, 0) &= (x_1, 1 - x_1 - y_1, 0), \\ \sigma C_6^3(x_1, x_2, 0) &= (1 - x_1 - x_2, x_2, 0), \\ \sigma C_6^{-1}(x_1, x_2, 0) &= (x_2, x_1, 0). \end{aligned} \quad (\text{D5})$$

In order that the bond goes back after some translations, it's straightforward to find out all the consistent conditions on such a bond:

$$\begin{aligned} T_2^{-1} \sigma C_6 : \langle -2x, x, 0 \rangle &\rightarrow \langle -2x, x, 0 \rangle \\ T_2^{-1} \sigma C_6 : \langle 0, x, 0 \rangle &\rightarrow \langle 0, x, 0 \rangle^\dagger \\ T_1^{-1} \sigma C_6^3 : \langle x, -2x, 0 \rangle &\rightarrow \langle x, -2x, 0 \rangle \\ T_1^{-1} \sigma C_6^3 : \langle x, 0, 0 \rangle &\rightarrow \langle x, 0, 0 \rangle^\dagger \\ \sigma C_6^{-1} : \langle x, x, 0 \rangle &\rightarrow \langle x, x, 0 \rangle \\ T_1^x T_2^{-x} \sigma C_6^{-1} : \langle x, -x, 0 \rangle &\rightarrow \langle x, -x, 0 \rangle^\dagger \end{aligned} \quad (\text{D6})$$

for  $\forall x \in \mathbb{Z}$ .

## 3. Conditions on a bond connecting different sublattices: $s = 1$

In the  $s = 1$  case, such a bond connects different sublattices. So only an even number of reflection and rotation, *i.e.*  $\nu_\sigma + \nu_{C_6} = 0 \pmod 2$  might transform the bond to itself, while an odd number of reflection and rotation, *i.e.*  $\nu_\sigma + \nu_{C_6} = 1 \pmod 2$  can transform the bond  $\langle x_1, x_2, 1 \rangle$  into its Hermitian conjugate  $\langle x_1, x_2, 1 \rangle^\dagger$ .

It's straightforward to obtain the following conditions on the bond  $\langle x_1, x_2, 1 \rangle \equiv u_{(x_1, x_2, 1)(0, 0, 0)}$ :

$$\begin{aligned} \sigma : \langle -2x, x, 1 \rangle &\rightarrow \langle -2x, x, 1 \rangle^\dagger \\ \sigma C_6^{-1} : \langle x + 1, x, 1 \rangle &\rightarrow \langle x + 1, x, 1 \rangle \\ T_1^{-2x-2} T_2^{x+1} \sigma C_6^{-2} : \langle -2x - 1, x, 1 \rangle &\rightarrow \langle -2x - 1, x, 1 \rangle^\dagger \\ T_1^{x-1} T_2^{x-2} C_6^3 : \langle x_1, x_2, 1 \rangle &\rightarrow \langle x_1, x_2, 1 \rangle^\dagger \\ T_1^{-1} \sigma C_6^3 : \langle x, -2x, 1 \rangle &\rightarrow \langle x, -2x, 1 \rangle \\ T_1^{x-1} T_2^{x-1} \sigma C_6^2 : \langle x + 1, x, 1 \rangle &\rightarrow \langle x + 1, x, 1 \rangle^\dagger \\ T_2^{-1} \sigma C_6 : \langle -2x - 1, x, 1 \rangle &\rightarrow \langle -2x - 1, x, 1 \rangle \end{aligned} \quad (\text{D7})$$

for  $\forall x, x_1, x_2 \in \mathbb{Z}$ .

## 4. An example: mean-field ansatz $\{u_{ij}\}$ of $Z_2$ spin liquids near u-RVB state

To demonstrate the above consistent conditions, let's take a look at how they determine the mean-field ansatz  $\{u_{ij}\}$  of any  $Z_2$  spin liquid near u-RVB state, with PSG generators (C2).

Considering time reversion  $\mathbf{T}$  we immediately have

$$g_{\mathbf{T}} \langle x_1, x_2, s \rangle = -(-1)^s \langle x_1, x_2, s \rangle g_{\mathbf{T}} \quad (\text{D8})$$

In other words, the bond connecting two sites belonging to the same (different) sublattice(s) anti-commutes(commutes) with  $g_{\mathbf{T}}$ .

For the nearest neighbor (n.n.) bond  $u_\alpha \equiv \langle 0, 0, 1 \rangle$  we have  $x_1 = x_2 = 0, s = 1$ . Conditions (D7) and (D8) immediately lead to

$$\begin{aligned} [u_\alpha, g_{\mathbf{T}}] &= 0 \\ g_\sigma u_\alpha &= -u_\alpha^\dagger g_\sigma \\ g_1^{-1} g_{C_6}^3 u_\alpha &= -u_\alpha^\dagger g_1^{-1} g_{C_6}^3 \end{aligned} \quad (\text{D9})$$

For 2nd n.n. bond  $u_\beta \equiv \langle 0, 1, 0 \rangle$  we have  $x_1 = 0 = s, x_2 = 1$  and (D6), (D8) lead to

$$\begin{aligned} \{u_\beta, g_{\mathbf{T}}\} &= 0 \\ g_\sigma g_{C_6} u_\beta &= u_\beta^\dagger g_\sigma g_{C_6} \end{aligned} \quad (\text{D10})$$

For 3rd n.n. bond  $u_\gamma \equiv \langle 1, 0, 1 \rangle$  we have  $x_2 = 0, x_1 = s = 1$ . Conditions (D7) and (D8) lead to

$$\begin{aligned} [u_\gamma, g_{\mathbf{T}}] &= 0 \\ g_{C_6}^3 u_\gamma &= -u_\gamma^\dagger g_{C_6}^3 \\ g_\sigma g_{C_6}^{-1} u_\gamma &= u_\gamma g_\sigma g_{C_6}^{-1} \end{aligned} \quad (\text{D11})$$

Constraints on further neighbors: *e.g.* 4th n.n.  $\langle 0, 1, 1 \rangle$ , 5th n.n.  $\langle 1, 1, 0 \rangle$  and 6th n.n.  $\langle 2, 0, 0 \rangle$  can be similarly obtained.

## Appendix E: A search of gapped spin liquids near the u-RVB state

In appendix C we showed that there are at most 24  $Z_2$  spin liquids around the u-RVB state, which are likely to connect with a semimetal through a continuous phase transition. In this section we search for those states with spectral gaps among the 24 spin liquid ansatz. In the end we find out most of the 24 states are gapless. More specifically, they cannot open up a mass gap through any perturbation around the u-RVB state, which has two graphenelike Dirac cones in the 1st Brillouin zone. It turns out that only 4 of them, *i.e.* #16, #17, #19 and #22 in TABLE I, are gapped spin liquids near the u-RVB state.

### 1. Symmetry-allowed masses in a graphenelike u-RVB state

We start from the low-energy effective Hamiltonian of the u-RVB state, which is described by a massless 8-component Dirac equation. These 8 components contain 2 spin indices (labeled by Pauli matrices  $\{\tau^i\}$ ), 2 sublattice indices (labeled by Pauli matrices  $\{\mu^i\}$ ) and 2 valley indices (labeled by Pauli matrices  $\{\nu^i\}$ ). Just like graphene, the two valleys are located at  $\mathbf{K}$  and  $\mathbf{K}'$ , *i.e.* the vertices in the honeycomb-shaped 1st Brillouin zone. Following the convention shown in FIG. 1, the momentum of these two cones are  $\mathbf{K} = \frac{4\pi}{3}\vec{b}_1 + \frac{2\pi}{3}\vec{b}_2$  and  $\mathbf{K}' = \frac{2\pi}{3}\vec{b}_1 + \frac{4\pi}{3}\vec{b}_2$  respectively, where  $\{\vec{b}_1 = (\sqrt{3}, -1)/\sqrt{3}a, \vec{b}_2 = (0, 2)/\sqrt{3}a\}$  are the reciprocal lattice vectors corresponding to lattice vectors  $\{\vec{a}_1 = (a, 0), \vec{a}_2 = (1, \sqrt{3})a/2\}$ .

Expanding the mean-field Hamiltonian of a u-RVB state with  $u_\alpha = i\tau^0$  (here  $\mathbf{k} = \frac{2}{\sqrt{3}a}(k_x, k_y) = k_1\vec{b}_1 + k_2\vec{b}_2$ )

$$H_{uRVB} = i(\psi_{\mathbf{k},A}^\dagger, \psi_{\mathbf{k},B}^\dagger) \cdot \begin{bmatrix} 0 & -\tau^0(1 + e^{-ik_2} + e^{i(k_1-k_2)}) \\ \tau^0(1 + e^{ik_2} + e^{i(k_2-k_1)}) & 0 \end{bmatrix} \cdot \begin{pmatrix} \psi_{\mathbf{k},A} \\ \psi_{\mathbf{k},B} \end{pmatrix}$$

around  $\mathbf{K}$  and  $\mathbf{K}'$  we immediately obtain the Dirac equations

$$H_{\mathbf{K}} = (\psi_{\mathbf{k},A}^\dagger, \psi_{\mathbf{k},B}^\dagger) \begin{bmatrix} 0 & \tau^0(k_y + ik_x) \\ \tau^0(k_y - ik_x) & 0 \end{bmatrix} \begin{pmatrix} \psi_{\mathbf{k},A} \\ \psi_{\mathbf{k},B} \end{pmatrix}$$

$$H_{\mathbf{K}'} = (\psi_{\mathbf{k}',A}^\dagger, \psi_{\mathbf{k}',B}^\dagger) \begin{bmatrix} 0 & \tau^0(k'_y - ik'_x) \\ \tau^0(k'_y + ik'_x) & 0 \end{bmatrix} \begin{pmatrix} \psi_{\mathbf{k}',A} \\ \psi_{\mathbf{k}',B} \end{pmatrix}$$

Defining the following 8-component spinor:

$$\Psi^T \equiv (\psi_{\mathbf{k},A}^T, \psi_{\mathbf{k},B}^T, \psi_{\mathbf{k}',B}^T, \psi_{\mathbf{k}',A}^T) \quad (\text{E1})$$

we can write the above effective Hamiltonian of u-RVB state as

$$H = \Psi^\dagger \mu^3 (\mu^2 \partial_x + \mu^1 \partial_y) \otimes \tau^0 \otimes \nu^0 \Psi \quad (\text{E2})$$

Therefore only those mass terms  $M = \mu^3 \otimes \tau^a \otimes \nu^b$ ,  $a, b = 0, 1, 2, 3$  satisfy that  $\{H, \Psi^\dagger M \Psi\} = 0$  so that a mass gap can be opened in the Dirac spectrum. In the following we study how the mass term changes under the action of symmetry transformation such as spin rotations, time reversal  $\mathbf{T}$  and translations  $T_1, T_2$  *etc.* The physical symmetry of a spin liquid state realized by mean-field ansatz only allow those masses that are invariant under the corresponding PSG. If a PSG already rules out all possible mass terms  $M = \mu^3 \otimes \tau^a \otimes \nu^b$ ,  $a, b = 0, 1, 2, 3$ , we conclude the corresponding spin liquid realized by mean-field ansatz is gapless.

First we work out the transformation rules of Dirac spinor  $\Psi$  and  $M$  under a PSG. We focus on the 24 PSG's near the u-RVB state with the form (C2) as summarized in TABLE I.

#### a. Spin rotations

It's straightforward to show that a spin rotation along  $\hat{z}$ -axis by  $2\theta$  angle is realized by

$$\Psi \rightarrow e^{i\theta} \Psi \quad (\text{E3})$$

while a spin rotation along  $\hat{y}$ -axis by  $\pi$  angle is realized by

$$\Psi \rightarrow i\tau^2 \mu^1 \nu^1 \Psi^* \quad (\text{E4})$$

Apparently  $S_z$  rotations leave the mass term invariant, while under  $S_y$  rotations by  $\pi$  the mass term transforms in the following way

$$M \rightarrow -\mu^1 \otimes \nu^1 \otimes \tau^2 M^T \tau^2 \otimes \mu^1 \otimes \nu^1 \quad (\text{E5})$$

Since the mass term is invariant under spin rotations, its allowed form as seen from the above constraint can only be

$$M_A^{(a)} = \mu^3 \otimes \nu^3 \otimes \tau^a, \quad a = 1, 2, 3 \quad (\text{E6})$$

or

$$M_B^{(b)} = \mu^3 \otimes \nu^b \otimes \tau^0, \quad b = 0, 1, 2 \quad (\text{E7})$$

#### b. Time reversal $\mathbf{T}$

Since a mean-field bond  $u_{ij}$  becomes  $-(-1)^{s_i} g_T u_{ij} g_T^\dagger (-1)^{s_j}$  under the time reversal transformation in a PSG (C2), clearly  $\mathbf{T}$  is realized by

$$\Psi \rightarrow g_T^\dagger \otimes \mu^3 \otimes \nu^3 \Psi$$

$$M \rightarrow -M \quad (\text{E8})$$

so the mass term is invariant under time reversal  $\mathbf{T}$  if

$$M = -g_T \otimes \mu^3 \otimes \nu^3 M g_T^\dagger \otimes \mu^3 \otimes \nu^3 \quad (\text{E9})$$

10 spin liquids near the u-RVB state, *i.e.* #1-#10 in TABLE I has  $g_T = \tau^0$ . In these cases, mass terms  $M_A^{(a)}$ ,  $a = 1, 2, 3$  will violate transformation rule (E9), and the only allowed masses are  $M_B^{(1)}, M_B^{(2)}$ .

The other 14 spin liquids around u-RVB state (#11-#24 in TABLE I) are characterized by  $g_T = i\tau^3$ . In this case the allowed masses are  $M_B^{(1)}, M_B^{(2)}$  and  $M_A^{(1)}, M_A^{(2)}$ .

#### c. Translations $T_1, T_2$

Under translations  $T_1, T_2$  in a PSG (C2) the 8-component spinor changes as

$$T_1 : \quad \Psi \rightarrow e^{-i\frac{2\pi}{3}\nu^3} \otimes g_1^\dagger \Psi,$$

$$T_2 : \quad \Psi \rightarrow e^{i\frac{2\pi}{3}\nu^3} \otimes g_2^\dagger \Psi. \quad (\text{E10})$$

since  $\mathbf{K} \cdot \vec{a}_{1,2} = \mp \frac{2\pi}{3}$  and  $\mathbf{K}' \cdot \vec{a}_{1,2} = \pm \frac{2\pi}{3}$ . In order for the mass term to be invariant

$$\begin{aligned} M &= e^{i\frac{2\pi}{3}\nu^3} \otimes g_1 M e^{-i\frac{2\pi}{3}\nu^3} \otimes g_1^\dagger \\ &= e^{-i\frac{2\pi}{3}\nu^3} \otimes g_2 M e^{i\frac{2\pi}{3}\nu^3} \otimes g_2^\dagger \end{aligned} \quad (\text{E11})$$

the symmetry-allowed masses can only be:

$$\begin{aligned} &M_B^{(0)} \text{ and } M_A^{(a)}, \quad a = 1, 2, 3 \text{ if } g_1 = g_2 = \tau^0; \\ &M_B^{(0)} \text{ and } M_A^{(3)} \text{ if } g_1 = g_2^{-1} = e^{i2\pi/3\tau^3}; \\ &M_B^{(0)} \text{ for the special case \#10 in TABLE I.} \end{aligned}$$

Combining conditions (E9) and (E11) we can see that  $\{M_B^{(b)}, b = 0, 1, 2\}$  are not allowed by symmetry in any of the 24 spin liquids near u-RVB state. In the following study we will focus on masses  $M_A^{(a)}$ ,  $a = 1, 2, 3$ .

#### d. Reflection $\sigma$

Similar to time reversal  $\mathbf{T}$ , under reflection along  $\hat{x}$ -axis the spinor transforms as

$$\Psi \rightarrow \mu^1 \cdot g_\sigma^\dagger \otimes \mu^3 \otimes \nu^3 \Psi = -i g_\sigma^\dagger \otimes \mu^2 \otimes \nu^3 \Psi \quad (\text{E12})$$

The mass term is invariant under reflection  $\sigma$  if

$$M = g_\sigma \otimes \mu^2 \otimes \nu^3 M g_\sigma^\dagger \otimes \mu^2 \otimes \nu^3 \quad (\text{E13})$$

The symmetry-allowed masses are:

$$\begin{aligned} &\text{none if } g_\sigma = \tau^0; \\ &M_A^{(a)}, \quad a \neq b \text{ if } g_\sigma = i\tau^b. \end{aligned}$$

#### e. $\pi/3$ rotation $C_6$

Under  $C_6$ , *i.e.* a rotation by  $\pi/3$  the spinor transforms as

$$\Psi \rightarrow g_{C_6}^\dagger \otimes e^{i\frac{5\pi}{6}\mu^3} \otimes \left(\frac{\sqrt{3}}{2}\nu^1 - \frac{1}{2}\nu^2\right) \Psi \quad (\text{E14})$$

The mass term is invariant under reflection  $\sigma$  if

$$\begin{aligned} M &= g_{C_6} \otimes e^{-i\frac{5\pi}{6}\mu^3} \otimes \left(\frac{\sqrt{3}}{2}\nu^1 - \frac{1}{2}\nu^2\right) \cdot M \\ &\cdot g_{C_6}^\dagger \otimes e^{i\frac{5\pi}{6}\mu^3} \otimes \left(\frac{\sqrt{3}}{2}\nu^1 - \frac{1}{2}\nu^2\right) \end{aligned} \quad (\text{E15})$$

The symmetry-allowed masses are:

$$\begin{aligned} &\text{none if } g_{C_6} = \tau^0, \quad e^{i\theta\tau^{1,3}} \text{ with } \theta \neq 0 \pmod{\pi/2}; \\ &M_A^{(a)}, \quad a \neq b \text{ if } g_{C_6} = i\tau^b. \end{aligned}$$

## 2. Realizing the 4 gapped $Z_2$ spin liquids near the u-RVB state

Among all 24 spin liquids near the u-RVB states, it turns out that there are no symmetry-allowed masses for 20 of them. In other words, these 20 spin liquids cannot open up a mass gap through a perturbation around the

#	$u_\alpha$	$u_\beta$	$u_\gamma$	$u_\delta$	$u_\varepsilon$	9th n.n. $\langle 1, 2, 0 \rangle$
16	$i\tau^0$	$\{\tau^1, \tau^2\}$	$i\tau^0$	$i\tau^0$	$\{\tau^1, \tau^2\}$	...
17	$i\tau^0$	$\tau^2$	$i\tau^0$	$\{i\tau^0, \tau^3\}$	...	...
19	$\{i\tau^0, \tau^3\}$	$\{\tau^1, \tau^2\}$	...	...	...	...
22	$i\tau^0$	$\tau^2$	$i\tau^0$	$i\tau^0$	$\tau^2$	$\{\tau^1, \tau^2\}$

TABLE II: Symmetry-allowed mean-field ansatz of the 4 possible gapped spin liquids near the u-RVB state. We follow the notations for mean-field bonds in appendix D. We only summarize the mean-field bonds that are necessary to realize a gapped  $Z_2$  spin liquid. Ellipsis represents those longer-range mean-field bonds unnecessary for a  $Z_2$  spin liquid, which are not listed in this table. Up to 3rd n.n. mean-field bonds  $\{u_\alpha, u_\beta, u_\gamma\}$ , only one  $Z_2$  spin liquid, *i.e.* #19 can be realized on a honeycomb lattice.

u-RVB state. Only the following 4 spin liquids near the u-RVB state can obtain an energy gap in the spectrum through adding a symmetry-allowed mass term:

#16 with two symmetry-allowed masses  $M_A^{(1,2)} = \mu^3 \otimes \nu^3 \otimes \tau^{1,2}$ ;

#17 with one symmetry-allowed mass  $M_A^{(2)} = \mu^3 \otimes \nu^3 \otimes \tau^2$ ;

#19 with one symmetry-allowed mass  $M_A^{(2)} = \mu^3 \otimes \nu^3 \otimes \tau^2$ ;

#22 with one symmetry-allowed mass  $M_A^{(2)} = \mu^3 \otimes \nu^3 \otimes \tau^2$ .

In fact, as summarized in TABLE II, all these 4 gapped spin liquids can be realized by mean-field ansatz  $\{u_{ij}\}$ , which satisfies consistent conditions from the corresponding PSG as discussed in appendix D. In the following we describe the mean-field ansatz for these 4 gapped  $Z_2$  spin liquids. In the end only one gapped  $Z_2$  spin liquid, *i.e.* #19 can be realized by a mean-field ansatz up to 3rd n.n. bonds.

#### a. $Z_2$ spin liquid #16: up to 5th n.n. bonds needed

The mean-field ansatz  $\{u_{ij}\}$  for  $Z_2$  spin liquid #16 is summarized in TABLE II, up to 5th n.n. bonds. The corresponding spin liquid has a  $Z_2$  gauge structure, if and only if  $[u_\beta, u_\varepsilon] \neq 0$ , so that the IGG of this mean-field ansatz is a  $Z_2$  group  $\{\pm\tau^0\}$ .

It's straightforward to check that 2nd n.n. bond  $u_\beta = \beta_1\tau^1 + \beta_2\tau^2$  open up a mass gap  $M \sim \mu^3 \otimes \nu^3 \otimes (\beta_1\tau^1 + \beta_2\tau^2) = \beta_1 M_A^{(1)} + \beta_2 M_A^{(2)}$ .

#### b. $Z_2$ spin liquid #17: up to 4th n.n. bonds needed

The mean-field ansatz  $\{u_{ij}\}$  for  $Z_2$  spin liquid #17 is summarized in TABLE II, up to 4th n.n. bonds. The corresponding spin liquid has a  $Z_2$  gauge structure, if and only if  $[u_\beta, u_\delta] \neq 0$ , so that the IGG of this mean-field ansatz is a  $Z_2$  group  $\{\pm\tau^0\}$ .



It's straightforward to check that 2nd n.n. bond  $u_\beta = \beta\tau^2$  open up a mass gap  $M \sim \beta\mu^3 \otimes \nu^3 \otimes \tau^2 = \beta M_A^{(2)}$ .

*c.  $Z_2$  spin liquid #19: up to 2nd n.n. bonds needed*

The mean-field ansatz  $\{u_{ij}\}$  for  $Z_2$  spin liquid #17 is summarized in TABLE II, up to 2nd n.n. bonds. The corresponding spin liquid has a  $Z_2$  gauge structure, if and only if  $u_\beta = \beta_1\tau^1 + \beta_2\tau^2$  with  $\beta_1, \beta_2 \neq 0$ , so that the IGG of this mean-field ansatz is a  $Z_2$  group  $\{\pm\tau^0\}$ . This is the only gapped  $Z_2$  spin liquid near the u-RVB state, that can be realized in a mean-field ansatz up to 3rd n.n. bonds.

It's straightforward to check that 2nd n.n. bond  $u_\beta = \beta_1\tau^1 + \beta_2\tau^2$  open up a mass gap  $M \sim \beta_2\mu^3 \otimes \nu^3 \otimes \tau^2 = \beta_2 M_A^{(2)}$ .

*d.  $Z_2$  spin liquid #22: up to 9th n.n. bonds needed*

The mean-field ansatz  $\{u_{ij}\}$  for  $Z_2$  spin liquid #17 is summarized in TABLE II, up to 9th n.n. bonds. The

corresponding spin liquid has a  $Z_2$  gauge structure, if and only if  $[u_\beta, u_9] \neq 0$ , so that the IGG of this mean-field ansatz is a  $Z_2$  group  $\{\pm\tau^0\}$ .  $u_9 \equiv \langle 1, 2, 0 \rangle$  is the 9th n.n. mean-field bond. In this  $Z_2$  spin liquid, the symmetry-allowed consistent mean-field bonds for 6th, 7th and 8th n.n. are:

$$\begin{aligned} u_6 &\equiv \langle 2, 0, 0 \rangle \propto \tau^2, \\ u_7 &\equiv \langle 2, 0, 1 \rangle \propto i\tau^0, \\ u_8 &\equiv \langle 0, 2, 1 \rangle \propto i\tau^0. \end{aligned}$$

It's straightforward to check that 2nd n.n. bond  $u_\beta = \beta\tau^2$  open up a mass gap  $M \sim \beta\mu^3 \otimes \nu^3 \otimes \tau^2 = \beta M_A^{(2)}$ .

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